EX: compute the tangent line ~~at~~ to the curve $F(t) = \langle 2\cos(t), 2\sin(t), 4\cos(2t)\rangle$ at a point $(-\sqrt{3}, 1, 2)$.

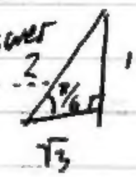
check sure

Sol: The tangent ~~vector~~ Function is: $r'(t) = \langle -2\sin(t), 2\cos(t), -8\sin(2t)\rangle$

To Find the time: Solve $\vec{r}(t) = \langle -\sqrt{3}, 1, 2\rangle$

i.e. $\begin{cases} 2\cos(t) = \sqrt{3} \\ 2\sin(t) = 1 \\ 4\cos(2t) = 2 \end{cases} \Rightarrow \begin{cases} \cos(t) = \frac{\sqrt{3}}{2} \\ \sin(t) = 1/2 \\ \cos(2t) = 1/2 \end{cases} \Rightarrow$

check to see if $\pi/6$ is the answer




try $t = \pi/6$

$2(\cos(\pi/6)) = \sqrt{3}$ ✓

$2\sin(\pi/6) = 1$ ✓ $\Rightarrow$ $\pi/6$ is a good answer

$4\cos(\pi/3) = 2$ ✓

$\therefore$ the tangent vector at $(-\sqrt{3}, 1, 2)$ is

$\vec{r}'(\pi/6) = \langle -2\sin(\pi/6), 2\cos(\pi/6), -8\sin(2\pi/6)\rangle$

$= \cancel{\langle -1, 2\sqrt{3}}$

$= \langle -2 \cdot 1/2, 2 \cdot \frac{\sqrt{3}}{2}, -8 \cdot \frac{\sqrt{3}}{2}\rangle$

$= \langle -1, \sqrt{3}, -4\sqrt{3}\rangle$

$\therefore$ the Desired tangent line has vector equation

$\vec{l}(t) = \vec{p} + t\vec{r}'(\pi/6) = \langle \sqrt{3}, 1, 2\rangle + t\langle -1, \sqrt{3}, -4\sqrt{3}\rangle$

$= \sqrt{3} t, 1 + \sqrt{3} t, 2 - (4\sqrt{3} t)\rangle$ ▨

## § 13.?: Arc length

Last time: the arc length of curve $\vec{r}(t)$ Between $t=a$ and $B$ is given by

$$S = \int_{t=a}^{B} \left| \vec{r}\,'(t) \right| \, Dt$$

From Calc II: the arc length was given by (for $\vec{r}(t) = \langle x(t), y(t) \rangle$ on $a \le t \le B$)

Same

arclength $\rightarrow$ $$S = \int_{t=a}^{B} \sqrt{\left(\frac{Dx}{Dt}\right)^2 + \left(\frac{Dy}{Dt}\right)^2} \, Dt = \int_{t=a}^{B} \sqrt{(x'(t))^2 + (y'(t))^2} \, Dt$$

EX: Compute the arclength of $\vec{r}(t) = \langle \cos(t), \sin(t), \ln(\cos(t)) \rangle$ on $0 \le t \le \pi/4$

Sol: $$S = \int_{t=a}^{B} \left| \vec{r}\,'(t) \right| Dt \qquad a=0 \quad B = \pi/4$$

$$\vec{r}\,'(t) = \left\langle -\sin(t), \cos(t), -\frac{\sin(t)}{\cos(t)} \right\rangle = \langle -\sin(t), \cos(t), -\tan(t) \rangle$$

$$\therefore \left|\vec{r}\,'(t)\right| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + (-\tan(t))^2} = \sqrt{\sin^2(t) + \cos^2(t) + \tan^2(t)}$$

$$= \sqrt{1 + \tan^2(t)} = \sqrt{\sec^2(t)} = |\sec(t)|$$

On $0 \le t \le \pi/4$, $\sec(t) \ge 0$, so $|\vec{r}\,'(t)| = \sec(t)$ on $0 \le t \le \pi/4$.

$$\therefore S = \int_{t=a}^{B} |\vec{r}\,'(t)| \, Dt = \int_{t=0}^{\pi/4} \sec(t) \, Dt.$$

(At this point, Chris has an offstage monologue to try and remember math he learned a while ago.)

$$= \Big[ \ln|\sec(t) + \tan(t)| \Big]_{t=0}^{\pi/4}$$

$$= \ln|\sec(\pi/4) + \tan(\pi/4)| - \ln|\sec(0) + \tan(0)|$$

$$= \ln|\sqrt{2} + 1| - \ln|1 + 0|$$

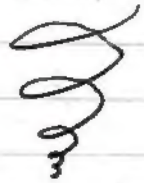
(Chris hates $\ln(1)$, please replace it with $0$)

$$= \ln(1 + \sqrt{2}) \quad \blacksquare$$

EX: Compute the arclength of $\vec{r}(t) = \langle 3\cos(t), 3\sin(t), t^2 \rangle$ on $2 \le t \le 10$

Sol: $S = \int_{t=\alpha}^{\beta} |\vec{r}\,'(t)|\, dt$  $\alpha = 2$  $\beta = 10$

looks like this



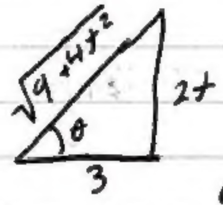
$\vec{r}\,'(t) = \langle -3\sin(t), 3\cos(t), 2t \rangle$.

$|\vec{r}\,'(t)| = \sqrt{(-3\sin(t))^2 + (3\cos(t))^2, (2t)^2}$

$\left( \begin{matrix} \sqrt{1+1} \ne \sqrt{1} + \sqrt{1} \\ \sqrt{2} \ne 2 \end{matrix} \right.$
chris got upset over this $\Big)$

$= \sqrt{9(\sin^2(t) + \cos^2(t) + 4t^2}$
$= \sqrt{9+4t^2}$

$\therefore S \int_{t=\alpha}^{\beta} |\vec{r}\,'(t)|\, dt = \int_{t=2}^{10} \sqrt{9+4t^2}\, dt$




$= \int_{t=2}^{10} 3 \cdot \frac{\sqrt{9+4t^2}}{3} \cdot \frac{2}{3} \cdot \frac{3}{2}\, dt.$

coordinate change

$\frac{2t}{3} = \tan(\theta)$

$\frac{2}{3}\, dt = \sec^2(\theta)\, d\theta$

$\frac{\sqrt{9+4t^2}}{3} = \sec(\theta)$

$= 9/2 \int_{t=2}^{10} \sec(\theta) \sec^2(\theta)\, d\theta$

$= 9/2 \int_{t=2}^{10} \sec^3(\theta)\, d\theta$

to compute $\int \sec^3(\theta)\, d\theta$:

$\int \sec^3(\theta)\, d\theta = \int \sec^2(\theta)\, d\theta = \sec(\theta)(1 + \tan^2(\theta))\, d\theta$

$= \int \sec(\theta) + \int \sec(\theta) \int \tan^2(\theta)\, d\theta$

$= \ln|\sec(\theta) + \tan(\theta)| + \sec(\theta) \tan^2(\theta)\, d\theta.$

$= \int \sec(\theta) \tan^2(\theta)\, d\theta$  $u = \tan(\theta)$  $dv = \sec(\theta)\tan(\theta)\, d\theta$
$du = \sec^2(\theta)\, d\theta$  $v = \sec(\theta)$

$= \int u\, dv = uv - \int v\, du$

$= \sec(\theta)\tan(\theta) - \int \sec(\theta) \sec^2(\theta)\, d\theta$

$= \sec(\theta)\tan(\theta) - \int \sec^3(\theta)\, d\theta$

$\therefore \int \sec^3(\theta)\, d\theta$
$= \ln|\sec(\theta) + \tan(\theta)|$
$+ \sec(\theta)\tan(\theta) - \int \sec^3(\theta)\, d\theta$

SO $2\int \sec^3(\theta) = \ln|\sec(\theta) + \tan(\theta)| + \sec(\theta)\tan(\theta) + C$

$\therefore \int \sec^3(\theta)\,d\theta$

$= \frac{1}{2}\left(\ln|\sec(\theta) + \tan(\theta)| + \sec(\theta) + \tan(\theta)\right) + C$

("C'mon guys, this is not that bad" - Chris)

Hence $\int = \frac{9}{2}\int_{t=2}^{10} \sec^3(\theta)\,d\theta$

$= \frac{9}{2}\left[\frac{1}{2}\left(\ln|\sec(\theta) + \tan(\theta)| + \sec(\theta)\ \tan(\theta)\right)\right]_{t=2}^{10}$

$= \frac{9}{4}\left[\ln\left|\frac{\sqrt{9+4t^2}}{3} + \frac{2t}{3}\right| + \frac{\sqrt{9+4t^2}}{3}\cdot\frac{2t}{3}\right]_{t=2}^{10}$

$= \frac{9}{4}\left(\ln\left|\frac{\sqrt{409}}{3} + \frac{20}{3}\right| + \frac{20}{9}\sqrt{409} - \ln\left|\frac{5}{3} + \frac{4}{3}\right| - \frac{5\cdot 4}{9}\right)$

$= \frac{9}{4}\left(\ln\left|\frac{\sqrt{409} + 20}{3}\right| + \frac{20}{9}\sqrt{409} - \ln(3) - \frac{20}{9}\right).$ $\blacksquare$

$= 5\left(\sqrt{409} - 1\right) + \frac{9}{4}\ln\left|\frac{20 + \sqrt{409}}{9}\right|$ $\blacksquare$ again.

(about 7 minutes of break where the entire class talked with Chris about his lack of memory)

The arc length of a curve is a natural choice for parameter.
I.e. we would like to parameterize $\vec{r}(t)$ so that at time $t=s$ the arc length (measured from some fixed point) is exactly $s$...


Define the arc length function for a parameterization by:

is the arc length $\sim$ $S(\beta) = \displaystyle\int_{t=a}^{\beta} |\vec{r}\,'(t)| \, dt.$ arc length function

By FTC $S'(\beta) = |\vec{r}\,'(\beta)|$

Moreover, $s$ is an increasing function provided $|\vec{r}\,'(\beta)| \neq 0$ for all $\beta$

$s$ is strictly increasing... means its injective, passes horizontal line test

On the next episode: this guarantees a unit speed parameterization of $\vec{r}(t)$

analogue for space curves
I.e. guarantees smoothness

(Chris wants review questions for Wednesday's class in prep for Friday's exam)